TO: Users of the ACTEX Review Seminar on DVD for SOA Exam FM/CAS Exam 2

FROM: Richard L. (Dick) London, FSA

Dear Students,

Thank you for purchasing the DVD recording of the ACTEX Review Seminar for SOA Exam FM (CAS Exam 2). This version is intended for the exam offered in May 2007 and thereafter. The purpose of this memo is to provide you with an orientation to the Interest Theory and elementary finance components of this seminar. A two-page summary of the topics covered in the seminar is attached to this cover memo. Although the seminar is organized independently of any particular textbook, the summary of topics shows where each topic is covered in the textbook *Mathematics of Investment and Credit* (Third Edition), by Samuel A. Broverman.

Interest Theory is a topic that is best taught, and understood, in a conceptual (rather than mathematical) framework. The amount of mathematics involved in the topic is quite light, being little more than the idea of summing a geometric series plus some very elementary calculus applications in connection with the force of interest and continuous annuities. However, although it is not very mathematically sophisticated, it is still a challenge to fully understand.

Throughout my review seminar that you are now viewing on DVD, you will hear me state that "There are no formulas, there are only concepts." Accordingly, the thrust of my review is to get the students to grasp these concepts. Once they are understood, problems can be solved with little difficulty because the mathematics is not complicated.

Actuarial exam review seminars always consist of an appropriate combination of review of theory and the working of practice problems. For topics that involve complex mathematics in the solution of problems, more practice problems should be worked by the instructor. For a topic like Interest Theory, however, with its elementary mathematics level, my educational philosophy is that our time together can be better spent by getting to understand the topic than by practicing our algebra.

Accordingly, you will find that a relatively small number of sample problems (about 35) are worked out for the group in this DVD. A copy of these questions, with detailed solutions, is attached to this cover memo for your reference. Of course you still need to do many, many, many practice problems as you prepare for your exam, and it is assumed that you have purchased one (or more) of the several study guides that have been prepared for that purpose.

With respect to the Finance Topics (see Section H of the Summary of Topics), please understand that these topics were added to the Interest Theory segment of Exam FM for the first time as of the May 2005 exam. Since a number of students might not have had previous course work in these Finance Topics, I have tried to give a more thorough presentation of the background theory, along with a few illustrative questions.

Good luck to you on your exam!

SUMMARY OF TOPICS

A. Interest Measurement

1. Accumulation Function (1.1.1)
2. Amount Function (1.1.3)
3. Effective Rate of Interest (1.1.1)
4. Effective Rate of Discount (1.4.1)
5. Simple Interest (1.1.2)
6. Compound Interest (1.1.1)
7. Present Value (1.2.1)
8. Nominal Rate Notation (1.3; 1.4.2)
9. Equivalence of Rates
10. Force of Interest (1.5)

B. Basic Concepts

1. Equation of Value (1.2.2)
2. Unknown Time (1.2.2)
3. Unknown Interest Rate (1.2.2)
4. Basic Calculator Usage

C. Level Payment Annuities

1. Annuity-Immediate (2.1.1 – 2.1.2)
2. Annuity-Due (2.1.3)
3. Perpetuity (2.1.2 – 2.1.3)
4. Unknown Time (Balloon and Drop) (2.2.3)
5. Unknown Interest Rate (Calculator Usage) (2.2.4)

D. General Annuities

1. General Approach for ICP ≠ PP (2.2.1)
2. m^{thly} Annuities (2.2.1)
3. Continuous Annuities (2.2.2)
4. Varying Annuities
    a. Geometic (2.3.1)
    b. Arithmetric (2.3.2)
    c. Continuous Varying (2.3.2)

E. Yield Rate Determination

1. General Yield Rate Equation (5.1.1)
2. Uniqueness (5.1.2)
3. Project Evaluation (5.1.3)
4. Book Value vs. Market Value (2.4.4)
5. Effect of Reinvestment (2.4.1)
6. Sinking Fund Valuation (2.4.5)
7. Dollar-Weighted Yield (5.2.1)
8. Time-Weighted Yield (5.2.2)
9. Investment Year / Portfolio Rates (5.3.1)
10. Continuous Model (5.3.3)

F. Loan Repayment

1. Accumulation Method (Zero-Coupon-Bond Model)
2. Interest-Only Method (Par Bond Model or Sinking Fund Model)
3. Amortization Method (3.1.1)
4. Outstanding Balance (3.1.3 – 3.1.4)
5. Amortization Schedules (3.1.2)
6. Other Properties (3.1.5; 3.2)
    a. Non-Level Interest
    b. Varying Payments
7. Sinking Funds (3.3)

G. Bonds

1. Terminology and Notation (4.1)
2. Bond Price (4.1.1)
    a. Basic Formula
    b. Alternate Formula (Premium Formula)
    c. Makeham's Formula
3. Premium and Discount (4.1.1)
4. Amortization and Accrual (4.2)
5. Off-Coupon-Date Values (4.1.2)
    a. Price
    b. Price-plus-accrued
6. Yield Rate Determination (Calculator Usage)
7. Callable Bonds (4.3.1)

H. Finance Topics

1. Theoretical Stock Prices (8.2.1)
2. Survey of Modern Financial Instruments
   a. Mutual Funds (8.2.4)
   b. CD's (8.3.1)
   c. Money Market Funds (8.3.2)
   d. Mortgage Backed Securities (8.3.3)
3. Effects of Inflation (1.6)
4. Yield Curves (Term Structure of Rates)
   a. Spot Rates (6.1)
   b. Forward Rates (6.3)
5. Duration (Macaulay Duration) (7.1)
6. Volatility (Modified Duration) (7.1)
7. Convexity (7.2)
8. Immunization (7.2)
9. Asset/Liability Matching (7.2)

ILLUSTRATIVE QUESTIONS

1. Jennifer deposits \$1000 into a bank account. The bank credits interest at nominal annual rate *j*, convertible semiannually, for the first seven years and nominal annual rate 2*j*, convertible quarterly, for all years thereafter. The accumulated value of the account at the end of 5 years is *X*. The accumulated value of the account at the end of 10.5 years is \$1980. Find *X*. (ANSWER: 1276.34)

2. A deposit of \$100 is made into a fund at time $t=0$. The fund pays interest at a nominal rate of discount $d^{(4)}$, compounded quarterly, for the first two years. Beginning at time $t=2$, interest is credited at a force of interest given by $\delta_t=(t+1)^{-1}$. At time $t=5$, the accumulated value of the fund is \$260. Find $d^{(4)}$. (ANSWER: .12906)

3. A deposit of *X* is made into a fund that pays an annual effective rate of interest of 6% for 10 years. At the same time, $\frac{X}{2}$ is deposited into another fund that pays an annual effective rate of discount of *d* for 10 years. The amounts of interest earned over the 10 years are equal for both funds. Find *d*. (ANSWER: .09049)

4. In Fund X money accumulates at force of interest $\delta_t = .01t+.10$, for $0<t<20$. In Fund Y money accumulates at annual effective rate *i*. An amount of \$1 is invested in each fund, and the accumulated values are the same at the end of 20 years. Find the value in Fund Y at the end of 1.5 years. (ANSWER: 1.35)

5. If $A(t) = Kt^2+Lt+M$, for $0<t<2$, with $A(0)=100$, $A(1)=110$, and $A(2)=136$, find the value of the force of interest at time $t=.50$. (ANSWER: .097)

6. Fund A accumulates at nominal rate 12%, convertible monthly. Fund B accumulates at force of interest $\delta_t=\frac{t}{6}$, for all *t*. At time $t=0$, \$1 is deposited in each fund. The accumulated values of the two funds are equal at time $n>0$, where *n* is measured in years. Find *n*. (ANSWER: 1.433)

7. On January 1, 1997, \$1000 is invested in a fund for which the force of interest at time *t* is given by $\delta_t=.10(t-1)^2$, where *t* is the number of years since January 1, 1997. Find the accumulated value of the fund on January 1, 1999. (ANSWER: 1068.94)

8. At $\delta_t=2(1+t)^{-1}$, payments of \$300 at $t=3$ and \$600 at $t=6$ have the same present value as payments of \$200 at $t=2$ and *X* at $t=5$. Find *X*. (ANSWER: 315.82)

9. Jeff and John have equal amounts of money to invest. John purchases a 10-year annuity-due with annual payments of \$2500 each. Jeff invests his money in a savings account earning 9% effective annual interest for two years. At the end of the two years, he purchases a 15-year immediate annuity with annual payments of *Z*. Both annuities are valued using an effective annual rate of 8%. Find the value of *Z*. (ANSWER: 2514.75)

10. Bill deposits money into a bank account at the end of each year. The deposit for year t is equal to $100t$, for $t = 1, 2, 3, \ldots$. The bank credits interest at effective annual rate i. \$500 of interest is earned in the account during the 11^{th} year. Find the value of i. (ANSWER: 7.26%)

11. A perpetuity pays 2 at the end of the 4^{th} year, 4 at the end of the 6^{th} year, 6 at the end of the 8^{th} year, and so on. Find the present value (at time 0) of the perpetuity using an effective annual interest rate of 10%. (ANSWER: 45.35)

12. Given that $\int_0^n \bar{a}_{\overline{t|}}\, dt = 100$, find the value of $\bar{a}_{\overline{n|}}$ in terms of n and δ. (ANSWER: $n - 100\delta$)

13. A continuously increasing annuity, with a term of n years, makes payment at annual rate t at time t. The force of interest is equal to $\frac{1}{n}$. Express the present value of the annuity as a function of n. (ANSWER: $.26424n^2$)

14. An immediate annuity makes monthly payments for 20 years. During the first year, each monthly payment is \$2000. Each year the monthly payments are 5% larger than in the previous year. Find the present value of the annuity at nominal interest rate $i^{(12)} = .06$. (ANSWER: 419,255)

15. A man deposits \$1000 into Fund A, which pays interest at the end of each six-month period at a certain nominal annual rate, convertible semiannually. These interest payments are taken from Fund A and immediately reinvested in Fund B, which earns interest at effective annual rate 6%. At the end of 15 years, the man has earned an effective annual yield rate of 7.56%. What is the nominal rate of interest being earned in Fund A? (ANSWER: 8.4%)

16. An investor pays P for an annuity that provides payments of \$100 at the beginning of each month for 10 years. These payments, when received, are immediately invested in Fund A, paying $i^{(12)} = .12$. The monthly interest payments from Fund A are reinvested in Fund B, earning $i^{(12)} = .06$. The investor's effective annual yield rate over the 10-year period is 8%. Find the value of P. (ANSWER: 9700)

17. You are given the following cash flow activity to an investment fund and the values of the fund on the dates shown:

Date	**Fund Value**	**Deposit**	**Withdrawal**
01/01	1,200		
03/31		100	150
04/01	1,400		
07/31		250	150
08/01	1,000		
10/31		350	150
11/01	1,600		
12/31	1,800		

Find the absolute value of the difference between the dollar-weighted and time-weighted rates of return. (ANSWER: .05939)

18. You are given the following cash flow activity to an investment fund, and the values of the fund *just before* the cash flows occur:

Date	Fund Value	Deposit	Withdrawal
01/01	1,000,000		
07/01	1,030,000		50,000
X	1,025,000	100,000	
12/31	1,150,000		

Given that the time-weighted and dollar-weighted rates of return are equal, find the date X. (ANSWER: November 15)

19. On January 1, \$90 is deposited into an investment account. On April 1, when the value in the fund has grown to X, a withdrawal of W is made. No other deposits or withdrawals are made throughout the year. On December 31, the value of the account is \$85. The dollar-weighted rate of return over the year is 20% and the time-weighted rate of return is 16%. Find the value of X. (ANSWER: 107.60)

The following table, which is used for both Questions 20 and 21, gives the pattern of investment year and portfolio interest rates over a three-year period, where $m = 2$ is the time after which the portfolio method is applicable.

Calendar Year of Original Investment, y	Investment Year Rate i_1^y	Investment Year Rate i_2^y	Portfolio Rates i^{y+2}	Calendar Year of Portfolio Rate, $y+2$
Z	9%	10%	11%	$Z+2$
$Z+1$	7%	8%		
$Z+2$	5%			

20. Frank invests \$1000 at the beginning of each of calendar years $Z, Z+1$ and $Z+2$. Find the amount of interest credited to Frank's account for calendar year $Z+2$. (ANSWER: 267.49)

21. An investment of \$1000 is made at the beginning of each of calendar years $Z, Z+1$ and $Z+2$. What is the average annual effective time-weighted rate of return for the three-year period? (ANSWER: 8.58%)

22. A loan charges interest at nominal annual rate $i^{(12)} = .24$. The loan is repaid with equal payments made at the end of each month for $2n$ months. The n^{th} payment is equally divided between interest and principal repaid. Find the value of n. (ANSWER: 34)

23. A loan of X is repaid by the sinking fund method over 10 years. The sinking fund earns annual effective rate 8%. The amount of interest earned in the sinking fund in the third year is \$85.57. Find the value of X. (ANSWER: 7449.58)

24. A loan of X, charging effective annual rate 8%, is to repaid by the end of 10 years. If the loan is repaid with a single payment at time $t=10$, the total amount of interest paid is \$468.05 more than it would be if the loan had been amortized by making ten equal annual payments. Find the value of X. (ANSWER: 700)

25. A loan, charging effective annual rate 8%, is repaid over 10 years with unequal annual payments. The first payment is X, and each subsequent payment is 10.16% greater than the one before it. The amount of interest contained in the first payment is \$892.20. Find the value of X. (ANSWER: 1100)

26. Tina buys a 10-year bond, of face (and redemption) amount \$1000, with 10% annual coupons at a price to yield 10% effective annual rate. The coupons, when received, are immediately reinvested at 8% effective annual rate. Immediately after receiving (and reinvesting) the 4^{th} coupon, Tina sells the bond to Joe for a price that will yield effective annual rate i to the buyer. The yield rate that Tina actually earns on her investment is 8% effective annual rate. Find the value of i. (ANSWER: 12.2%)

27. John purchases a 10-year bond of face amount \$1000. The bond pays coupons at annual rate 8%, payable semiannually. The redemption value is R. The purchase price is \$800, and the present value of the redemption value is \$301.51. Find R. (ANSWER: 800)

28. A 10-year bond has face value \$1000, but will be redeemed for \$1100. It pays semiannual coupons, and is purchased for \$1135 to yield 12%, convertible semiannually. The first coupon is X, and each subsequent coupon is 4% greater than the preceding coupon. Find the value of X. (ANSWER: 50)

29. The rate of interest is 8% and the rate of inflation is 5%. A single deposit is invested for 10 years. Let A denote the value of the investment at the end of 10 years, measured in time 0 dollars, and let B denote the value of the investment at the end of 10 years computed at the real rate of interest. Find the ratio of $\frac{A}{B}$. (ANSWER: 1.00)

30. Rework Question 29 assuming equal level deposits at the beginning of each year during the 10-year period instead of a single deposit. (ANSWER: .81995)

31. Consider the following table of spot rates:

Term of Investment	**Spot Rate**
1	7.00%
2	8.00
3	8.75
4	9.25
5	9.50

Find each of (a) the 1-year deferred, 2-year forward rate, and (b) the 2-year deferred, 3-year forward rate. (ANSWERS: 9.64%; 10.51%)

32. Find the duration of a common stock that pays dividends at the end of each year, if it is assumed that each dividend is 4% greater than the prior dividend and the effective rate of interest is 8%. (ANSWER: 27)

33. Find the convexity of the common stock described in Question 32. (ANSWER: 1250)

34. A financial institution accepts an $85,000 deposit from a customer on which it guarantees to pay 8% compounded annually for ten years. The only investments available to the institution are five-year zero-coupon bonds and preferred stocks, both yielding 8%. The institution develops an investment policy using the following reasoning:

(a) The duration of the five-year zero-coupon bond is 5.
(b) The duration of the preferred stock is 13.50.
(c) The duration of the obligation to the customer is 10.
(d) Taking the weighted-average of the durations, the amount invested in five-year zero-coupon bonds is chosen to be

$$\left(\frac{13.5-10}{13.5-5}\right)(85,000) = \$35,000$$

and the amount invested in preferred stock is chosen to be

$$\left(\frac{10-5}{13.5-5}\right)(85,000) = \$50,000.$$

Verify that this investment strategy is optimal under immunization theory, assuming the customer leaves the funds on deposit for the full ten-year period.

35. An insurance company accepts an obligation to pay 10,000 at the end of each year for two years. The company purchases a combination of the following two bonds at a total cost of X in order to exactly match its obligation:

(i) 1-year 4% annual coupon bond with a yield rate of 5%.
(ii) 2-year 6% annual coupon bond with a yield rate of 5%.

(a) Find the value of X. (ANSWER: 18,594)
(b) Find the face amounts of the 4% and 6% bonds. (ANSWER: 9071.12: 9433.96)

SOLUTIONS TO ILLUSTRATIVE QUESTIONS

1. We are given that

$$1000\left(1+\frac{j}{2}\right)^{14}\left(1+\frac{2j}{4}\right)^{14} = 1980,$$

or

$$\left(1+\frac{j}{2}\right)^{28} = 1.980,$$

so

$$\frac{j}{2} = (1.980)^{1/28}-1 = .0247.$$

Then

$$X = 1000(1.0247)^{10} = \underline{\underline{1276.34}}.$$

2. We are given

$$100\left(1-\frac{d^{(4)}}{4}\right)^{-8}\cdot e^{\int_2^5 1/t+1\,dt} = 260.$$

The integral evaluates to

$$e^{\ln(t+1)|_2^5} = e^{\ln(6/3)} = 2.$$

Then

$$\left(1-\frac{d^{(4)}}{4}\right)^{-8} = \frac{260}{(2)(100)} = 1.30,$$

which solves for

$$d^{(4)} = \left[1-(1.30)^{-1/8}\right](4) = \underline{\underline{.12906}}.$$

3. We are given $AV_{10} = X(1.06)^{10}$, so the amount of interest is

$$X\left[(1.06)^{10}-1\right] = .79085X.$$

We are also given $AV_{10} = \frac{X}{2}(1-d)^{-10}$, so the amount of interest is

$$\frac{X}{2}\left[(1-d)^{-10}-1\right] = .79085X,$$

since the interest amounts are the same. Then

$$(1-d)^{-10}-1 = 1.5817,$$

which solves for

$$d = 1-(2.5817)^{-1/10} = \underline{\underline{.09049}}.$$

4. In Fund X, $AV_{20} \;=\; e^{\int_0^{20}(.01t+.10)\,dt} \;=\; e^{\left(.005t^2+.10t\right)_0^{20}} \;=\; 54.59815.$

In Fund Y, $AV_{20} \;=\; (1+i)^{20} \;=\; 54.59815,$ so $i = .2214.$

Then $AV_{1.5} \;=\; (1.2214)^{1.5} \;=\; \underline{\underline{1.34985}}.$

5. The first condition implies $A(0) = M = 100,$ so $M = 100.$

The second condition implies $A(1) \;=\; K + L + 100 \;=\; 110,$ so $K + L = 10.$

The third condition implies $A(2) \;=\; 4K + 2L + 100 \;=\; 136,$ so $2K + L = 18.$

These equations solve for $K = 8$ and $L = 2,$ so $A(t) = 8t^2 + 2t + 100.$

Then

$$\delta_t \;=\; \frac{\frac{d}{dt}A(t)}{A(t)} = \frac{16t+2}{8t^2+2t+100},$$

and

$$\delta_{.50} \;=\; \frac{16(.50)+2}{8(.50)^2+2(.50)+100} \;=\; \underline{\underline{.09709}}.$$

6. We are given

$$AV_n^A \;=\; (1.01)^{12n} \;=\; e^{\int_0^n t/6\,dt} \;=\; AV_n^B.$$

Then $(1.01)^{12n} = e^{n^2/12}.$ Taking natural logs, we have

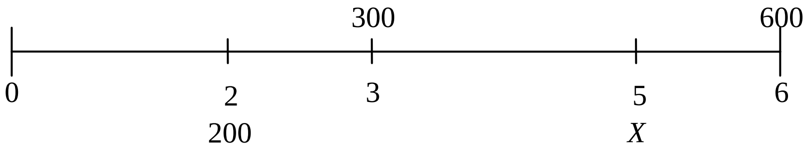
$$12n \cdot \ln(1.01) \;=\; (.00995)(12n) \;=\; \frac{n^2}{12},$$

or $n^2 = 1.433n,$ so $n = \underline{\underline{1.433}}.$

7. $$AV_2 \;=\; 1000e^{\int_0^2 .10(t-1)^2\,dt} \;=\; 1000 \cdot \exp\left[\frac{.10}{3}(t-1)^3\right]_0^2$$

$$= 1000 \cdot e^{.10/3[1-(-1)]} \;=\; 1000e^{.20/3} \;=\; \underline{\underline{1068.94}}.$$

8.



$$300e^{-\int_0^3 2(1+t)^{-1}\,dt} + 600e^{-\int_0^6 2(1+t)^{-1}\,dt} \;=\; 200e^{-\int_0^2 2(1+t)^{-1}\,dt} + X\cdot e^{-\int_0^5 2(1+t)^{-1}\,dt}$$

In general,

$$-\int_0^n 2(1+t)^{-1}\,dt \;=\; -2\cdot\ln(1+t)\,\Big|_0^n \;=\; \ln(1+n)^{-2}.$$

Then we have

$$300(1+3)^{-2} + 600(1+6)^{-2} \;=\; 200(1+2)^{-2} + X\,(1+5)^{-2},$$

or

$$\frac{300}{16} + \frac{600}{49} \;=\; \frac{200}{9} + \frac{X}{36},$$

which solves for $X \;=\; \underline{\underline{315.82.}}$

9. The amount each has to invest can be found as the present value of John's annuity:

$$PV \;=\; 2500\,\ddot{a}_{\overline{10}|.08} \;=\; 18{,}117.18.$$

Then Jeff invests an equal amount for two years at 9%, so he has

$$18{,}117.18(1.09)^2 \;=\; 21{,}525.03$$

at time $t=2$. This amount is then the present value of the immediate annuity, so

$$21{,}525.03 \;=\; Z\cdot a_{\overline{15}|.08}$$

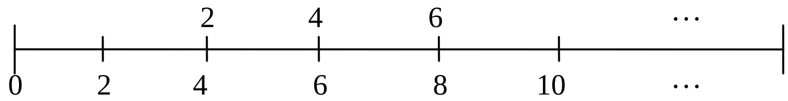
so $Z \;=\; \dfrac{21{,}525.03}{a_{\overline{15}|.08}} \;=\; \underline{\underline{2514.75}}.$

10. The balance in the account at time $t=10$ is $100(Is)_{\overline{10}|i}$, so the interest earned in the 11^{th} year is

$$500 \;=\; i\Big[100(Is)_{\overline{10}|i}\Big] \;=\; 100\Big(\ddot{s}_{\overline{10}|} - 10\Big),$$

from which we find $\ddot{s}_{\overline{10}|i} = 15$. We use a BA-35 calculator to directly find $\underline{\underline{i = 7.26\%.}}$

11.



The payments are made every two years, so we first find the effective interest rate over a two-year period. This is

$$j = (1.10)^2 - 1 = .21.$$

The value of d corresponding to this j is $d_j = \frac{.21}{1.21}$. The present value of the perpetuity at time $t = 2$ is

$$PV = 2\left(\frac{1}{j \cdot d_j}\right) = \frac{2}{(.21)\left(\frac{.21}{1.21}\right)} = 54.87528,$$

and the present value at time $t = 0$ is $54.87528(1.10)^{-2} = \underline{\underline{45.35}}$.

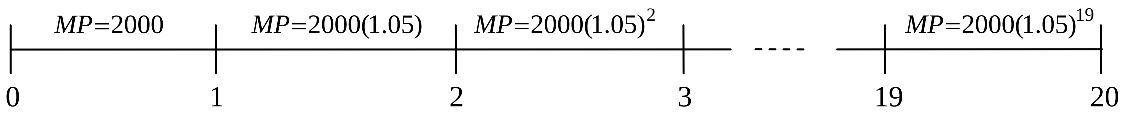
12. $\int_0^n \bar{a}_{\overline{t|}}\, dt = \int_0^n \frac{1-v^t}{\delta}\, dt = \frac{1}{\delta}\left[\int_0^n dt - \int_0^n v^t dt\right] = \frac{1}{\delta}\left[n - \bar{a}_{\overline{n|}}\right] = 100$.

Then $\bar{a}_{\overline{n|}} = \underline{\underline{n - 100\delta}}$.

13. $PV = \int_0^n t \cdot v^t\, dt = \int_0^n t \cdot e^{-\delta t}\, dt$, since $e^{\delta} = 1 + i$. In this case $\delta = \frac{1}{n}$, so we have

$$\begin{aligned}
PV &= \int_0^n \begin{array}{c|c} t \cdot & e^{-t/n}\, dt \\ \hline dt & \frac{e^{-t/n}}{-1/n} \end{array} \\
&= -t \cdot n \cdot e^{-t/n}\Big|_0^n + n \cdot \int_0^n e^{-t/n}\, dt \\
&= -n^2 \cdot e^{-1} + n\left(\frac{e^{-t/n}}{-1/n}\right)_0^n \\
&= -n^2 \cdot e^{-1} - n^2 \cdot e^{-1} + n^2 \\
&= n^2(1 - 2e^{-1}) = \underline{\underline{.26424n^2}}
\end{aligned}$$

14.



Note that the payment is level within each year, and then increases for the next year. The value of the first year's payments at time $t=1$ is

$$2000s_{\overline{12|}.005} = 24,671.00.$$

The value of the second year's payments at time $t=2$ is therefore 24,671(1.05), and so on. Then the overall present value is

$$PV \;=\; 24,671v + 24,671(1.05)v^2 + \cdots + 24,671(1.05)^{19}v^{20},$$

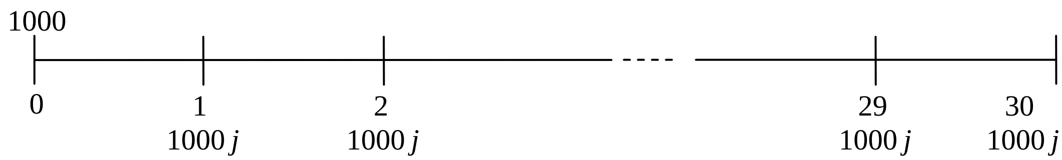
where time is measured in years.

The effective annual rate is $j \;=\; (1.005)^{12} - 1 \;=\; .06168.$ Then

$$\begin{aligned} PV &= \frac{24,671}{1+j}\left[1 + \frac{1.05}{1+j} + \left(\frac{1.05}{1+j}\right)^2 + \cdots + \left(\frac{1.05}{1+j}\right)^{19}\right] \\ &= \frac{24,671}{1+j}\left[\frac{1-\left(\frac{1.05}{1+j}\right)^{20}}{1-\left(\frac{1.05)}{1+j}\right)}\right] \\ &= \frac{24,671}{1.06168}\left(\frac{1-.80152}{1-.98899}\right) \;=\; \underline{\underline{418,912.}} \end{aligned}$$

There will be rounding errors in this problem.

15.



The interest payments from Fund A are 1000j each, where j is the effective semiannual rate. They are then invested in Fund B, which earns $k = (1.06)^{1/2} - 1 \;=\; .029563$ effective semiannually. After 15 years $(t=30)$, the value in Fund B is

$$1000j \cdot s_{\overline{30|}k} \;=\; 47,240.12j,$$

and the value in Fund A is still 1000. Then the equation of value to define the yield rate, known to be 7.56% effective annual, is

$$1000(1.0756)^{15} \;=\; 1000 + 47,240.12j,$$

which solves for $j = .041992$, and the nominal rate is $2j = \underline{\underline{.08399}}.$

16.

	P							
	0	1	2	3	4	$\cdots$	119	120
A:	100	100	100	100	100	$\cdots$	100	
B:		1	2	3	4	$\cdots$	119	120

Fund A earns .01 effective per month, so the interest is \$1 from each 100 invested in Fund A. Thus the deposits to Fund B are $1,2,\ldots,120$ at the end of each month.

At time $t=120$ the balance in Fund A is

$$(100)(120) = 12,000$$

and the balance in Fund B is

$$(Is)_{\overline{120}|.005} = \frac{\ddot{s}_{\overline{120}|.005} - 120}{.005} = 8,939.88.$$

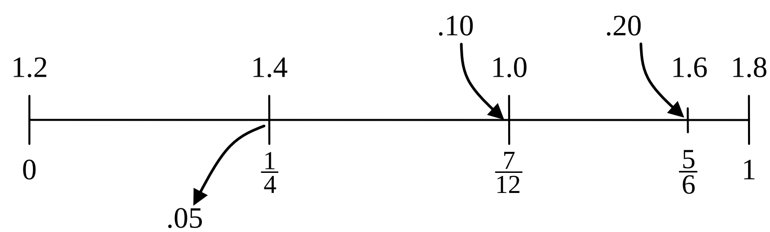
The equation of value to define the yield rate is then

$$P(1.08)^{10} = 12,000 + 8,939.88,$$

so

$$P = (12,000 + 8,939.88)(1.08)^{-10} = \underline{\underline{9699.24}}.$$

17.



[Note that deposits and withdrawals at the same time point can be netted.]

Time-weighted:

$$i_T = \left(\frac{1.45}{1.2}\right)\left(\frac{.9}{1.4}\right)\left(\frac{1.4}{1.0}\right)\left(\frac{1.8}{1.6}\right) - 1 = .22344$$

[Note that the actual time point values are not used.]

Dollar-weighted:

$$1.2(1+i) - .05(1+i)^{3/4} + .10(1+i)^{5/12} + .20(1+i)^{1/6} = 1.8$$

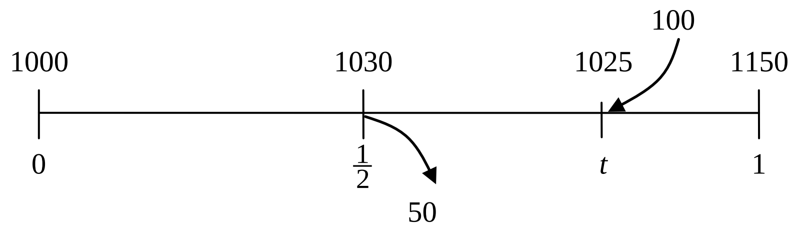
$$i\left[1.2 - (.05)(.75) + (.10)\left(\tfrac{5}{12}\right) + (.20)\left(\tfrac{1}{6}\right)\right] = 1.8 - 1.2 + .05 - .10 - .20$$

$$i_D = \frac{.35}{1.2375} = .28282.$$

[Note that the intermediate fund balances are not used.]

Then, $i_D - i_T = \underline{\underline{.05939}}$.

18. This time we are solving for an unknown date.



Recall that the time-weighted rate does *not* use the dates. Then

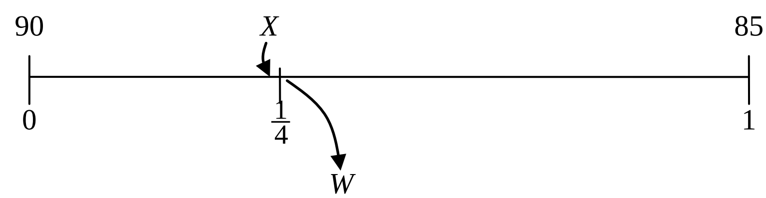
$$i_T = \left(\frac{1030}{1000}\right)\left(\frac{1025}{980}\right)\left(\frac{1150}{1125}\right) - 1 = .10124.$$

But we are given that $i_T = i_D$. Therefore we are not solving for i_D, but rather solving for the unknown time point t. Let $r = 1 - t$.

$$1000(1+i) - 50(1+i)^{1/2} + 100(1+i)^r = 1150$$
$$i\left[1000 - \tfrac{1}{2}(50) + 100r\right] = 1150 - 1000 + 50 - 100 = 100$$

Then $i = \frac{100}{975+100r} = .10124$, which solves for $r = .12751$. This is the fraction of year remaining as of date X. Then $365r = 46.5$ days remain, so date X is 46.5 days back from the end of the year, which is about $\underline{\underline{\text{November 15}}}$.

19.



Both X and W are unknown. But the dollar-weighted method does not use the value of X, so we use it to solve for W.

$$90(1+i) - W(1+i)^{3/4} = 85$$

$$i_D = \frac{85 - 90 + W}{90 - \frac{3}{4}W} = .20$$

$$W - 5 = 18 - .15W$$

$$W = 20$$

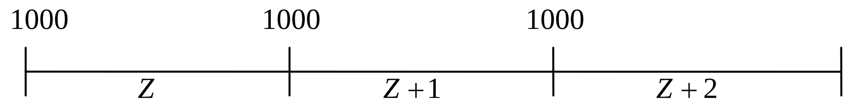
Then

$$1 + i_T = \left(\frac{X}{90}\right)\left(\frac{85}{X-20}\right) = 1.16$$

$$85X = 104.4X - 2088$$

$$X = \underline{\underline{107.63}}.$$

20.



The investment on 1/1/Z earns 9%, so the fund value on 12/31/Z is 1090.00.

In year $Z+1$, the investment from year Z earns 10% and the investment from year $Z+1$ earns 7%, so the fund value on $12/31/Z+1$ is $(1090)(1.10)+(1000)(1.07) = 2269.00$.

In year $Z+2$, the investment from year Z earns 11%, the investment from year $Z+1$ earns 8%, and the investment from year $Z+2$ earns 5%, so the fund value on $12/31/Z+2$ is

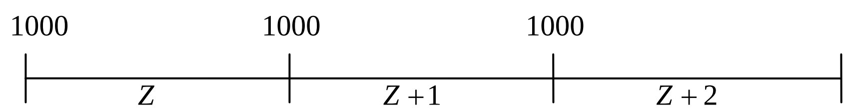
$$(1090)(1.10)(1.11)+(1070)(1.08)+(1000)(1.05) = 3536.49.$$

Then the time-weighted rate of return is

$$i_T = \left(\frac{1090}{1000}\right)\left(\frac{2269}{2090}\right)\left(\frac{3536.49}{3269.00}\right)-1 = .28018,$$

effective over the three-year period. The annual effective rate is $i=(1.28018)^{1/3}-1=\underline{\underline{.08582}}$.

21.



The balance in the account on $12/31/Z+1$ is

$$1000(1.09)(1.10)+1000(1.07) = 2269.00.$$

The balance in the account on $12/31/Z+2$ is

$$1000(1.09)(1.10)(1.11)+1000(1.07)(1.08)+1000(1.05) = 3536.49.$$

Then the growth during $Z+2$ is

$$3536.49-2269.00 = 1267.49,$$

of which 1000 comes from the $1/1/Z+2$ deposit, so the interest earned during calendar year $Z+2$ is $\underline{\underline{267.49}}$.

22. The effective monthly rate is $i=.02$ and the term of the loan is $2n$ months. Then

$$PR_n \;=\; P\cdot v^{2n-n+1}$$

and

$$I_n \;=\; P\left(1-v^{2n-n+1}\right).$$

But $PR_n = I_n$, so we have

$$v^{2n-n+1} \;=\; v^{n+1} \;=\; \frac{1}{2}.$$

Then $n+1 \;=\; \dfrac{\ln\frac{1}{2}}{\ln v} \;=\; \dfrac{\ln .50}{\ln(1.02)^{-1}} \;=\; 35,$ and $\underline{\underline{n=34}}$.

23. The sinking fund deposit is $\dfrac{X}{s_{\overline{10}|.08}}$, so the sinking fund balance after 2 payments (beginning of the third year) is

$$SFB_2 \;=\; \left(\frac{X}{s_{\overline{10}|.08}}\right)s_{\overline{2}|.08}.$$

The interest in the third year is $i\cdot SFB_2$, so we have $i\cdot SFB_2 \;=\; \dfrac{X}{s_{\overline{10}|.08}}\left[(1.08)^2-1\right] \;=\; 85.57,$ so

$X \;=\; \dfrac{85.57}{(1.08)^2-1}\cdot s_{\overline{10}|.08} \;=\; \underline{\underline{7449.61}}$.

24. Under the accumulation method, $TI \;=\; X\left[(1.08)^{10}-1\right]$.

Under the amortization method, $TI \;=\; 10P-X \;=\; \dfrac{10X}{a_{\overline{10}|.08}}-X$.

We are given

$$X\left[(1.08)^{10}-1\right] \;=\; 468.05+X\left[\frac{10}{a_{\overline{10}|.08}}-1\right], \text{ so } X \;=\; \frac{468.05}{(1.08)^{10}-\frac{10}{a_{\overline{10}|.08}}} \;=\; \underline{\underline{700}}.$$

25. The amount of interest in the first payment is $i\cdot L$, so we have $.08L = 892.20$, so $L = 11{,}152.50$. Then

$$\begin{aligned} 11{,}152.50 &= X\cdot v + X(1.1016)v^2 + x(1.1016)^2v^3 + \cdots + X(1.1016)^9v^{10} \\ &= \frac{X}{1.08}\left[1+\frac{1.1016}{1.08}+\left(\frac{1.1016}{1.08}\right)^2+\cdots+\left(\frac{1.1016}{1.08}\right)^9\right] \\ &= \frac{X}{1.08}\left[\frac{1-\left(\frac{1.1016}{1.08}\right)^{10}}{1-\frac{1.1016}{1.08}}\right] \;=\; 10.13843X, \end{aligned}$$

so $X = \dfrac{11{,}152.50}{10.13843} = \underline{\underline{1100}}$.

26. For the bond, $C = F = 1000$ and $g = r = i = .10$. Since $g = i$, then $P = C = 1000$ is the price that Tina invests at time $t = 0$.

At time 4, the reinvested coupons have accumulated to $100s_{\overline{4}|.08} = 450.61$. The price Joe pays for the bond is

$$P = 1000 + 1000(.10 - i)a_{\overline{6}|i} = 1000v_i^6 + 100a_{\overline{6}|i}.$$

Since Tina earns 8% then, for her, $1000(1.08)^4 = 450.61 + P$, so $P = 909.88 = 1000v_i^6 + 100a_{\overline{6}|i}$. Solving by calculator gives $\underline{\underline{i = 12.2\%}}$.

27. Coupons are paid semiannually, and each coupon is $.04(1000) = 40$. The price is given, so we have

$$P = 800 = R \cdot v_i^{20} + 40a_{\overline{20}|i}.$$

We are also given that $R \cdot v_i^{20} = 301.51$, so we have

$$800 = 301.51 + 40\, a_{\overline{20}|i},$$

so $a_{\overline{20}|i} = \dfrac{800 - 301.51}{40} = 12.46225$, which solves for $i = .05$. Then

$$R = \left[800 - 40(12.46225)\right](1.05)^{20} = \underline{\underline{800.00}}.$$

28. There are 20 semiannual coupons. The effective semiannual yield rate is .06. The present value of the coupons is

$$\begin{aligned} Xv + X(1.04)v^2 + X(1.04)^2v^3 + \cdots + X(1.04)^{19}v^{20} \\ &= \frac{X}{1.06}\left[1 + \frac{1.04}{1.06} + \left(\frac{1.04}{1.06}\right)^{20} + \cdots + \left(\frac{1.04}{1.06}\right)^{19}\right] \\ &= \frac{X}{1.06}\left(\frac{1 - \left(\frac{1.04}{1.06}\right)^2}{1 - \frac{1.04}{1.06}}\right) = 15.83985X. \end{aligned}$$

Then we have

$$1135 = 15.83985X + 1100v_{.06}^{20},$$

so

$$X = \frac{1135 - 1100(1.06)^{-20}}{15.83985} = \underline{\underline{50.00}}.$$

29. The accumulated value is $(1.08)^{10} = 2.15892.$ But this is measured in dollars valued 10 years after time 0, and is not equal to 2.15892 "time 0 dollars" because of inflation. The value in time 0 dollars is

$$A = \frac{2.15892}{(1.05)^{10}} = 1.32539,$$

which is $\left(\frac{1.08}{1.05}\right)^{10} = (1+i')^{10} = B,$ so the ratio $\frac{A}{B}$ is $\underline{\underline{1.00}}$.

30. In this case each deposit accumulates at 8% from its deposit date, but is then discounted back to time 0 at 5% to express the accumulated value in time 0 dollars. This produces

$$A = \frac{\ddot{s}_{\overline{10}|.08}}{(1.05)^{10}} = 9.60496.$$

Then $B = \ddot{s}_{\overline{10}|i'} = \ddot{s}_{\overline{10}|.02857} = 11.71402,$ so the ratio is $\frac{A}{B} = \underline{\underline{.81995}}$.

31. (a) A unit earning the current 1-year spot rate followed by the 1-year deferred 2-year forward rate will accumulate to the same result as using the current 3-year spot rate. That is, $(1.07)(1+f)^2 = (1.0875)^3,$ which solves for $f = .09636,$ or 9.64%.

(b) Similarly, accumulation at the current 2-year spot rate followed by the 2-year deferred 3-year forward rate is the same as accumulation at the current 5-year spot rate. Then $(1.08)^2(1+f)^3 = (1.095)^5,$ which solves for $f = .10512$, or $\underline{\underline{10.51\%}}$.

32. If the dividend at the end of the first year is D, then the present value of all payments is

$$\begin{aligned} P(i) &= D\left[(1+i)^{-1} + (1.04)(1+i)^{-2} + (1.04)^2(1+i)^{-3} + \cdots\right] \\ &= D(1+i)^{-1}\left[1 + \left(\frac{1.04)}{1+i}\right) + \left(\frac{1.04}{1+i}\right)^2 + \cdots\right] \\ &= D(1+i)^{-1}\left[\frac{1+i}{i-.04}\right] = D(i-.04)^{-1}. \end{aligned}$$

Then

$$P'(i) = -D(i-.04)^{-2},$$

so the volatility is

$$\bar{v} = -\frac{P'(i)}{P(i)} = (i-.04)^{-1}$$

and the duration, evaluated at 8%, is

$$\bar{d} = \bar{v}(1.08) = (.08-.04)^{-1}(1.08) = \underline{\underline{27}}.$$

33. From Question 32 we have $P(i) = D(i-.04)^{-1}$. Then

$P'(i) = -D(i-.04)^{-2}$ and $P''(i) = 2D(i-.04)^{-3}$, so

$$\bar{c} = \frac{P''(i)}{P(i)} = 2(i-.04)^{-2},$$

which, when evaluated at $i = .08$, gives $\bar{c} = \underline{\underline{1250}}$.

34. The cash flow on the assets (the investment plan) is $35{,}000(1.08)^5$ at time 5 (from the bond) and $(.08)(50{,}000) = 4000$ at each of times $1, 2, \ldots$ (from the stock). The cash flow on the liability (the deposit) is $85{,}000(1.08)^{10}$. The present value of the asset cash inflow, as a function of i, is

$$P_A(i) = 35{,}000(1.08)^5(1+i)^{-5} + 4000i^{-1},$$

and the present value of the liability cash outflow, also as a function of i, is

$$P_L(i) = 85{,}000(1.08)^{10}(1+i)^{-10}.$$

The first derivatives are

$$P_A'(i) = -(5)(35{,}000)(1.08)^5(1+i)^{-6} - 4000i^{-2}$$

and

$$P_L'(i) = -(10)(85{,}000)(1.08)^{10}(1+i)^{-11},$$

and the second derivatives are

$$P_A''(i) = (5)(6)(35{,}000)(1.08)^5(1+i)^{-7} + 8000i^{-3}$$

and

$$P_L''(i) = (10)(11)(85{,}000)(1.08)^{10}(1+i)^{-12}.$$

At 8% effective annual rate, we find $P_A(.08) = P_L(.08) = 85{,}000$. Furthermore,

$$\bar{v}_A = -\frac{P_A'(.08)}{P_A(.08)} = \frac{787{,}037.04}{85{,}000.00} = 9.25926$$

and

$$\bar{v}_L = -\frac{P_L'(.08)}{P_L(.08)} = \frac{787{,}037.04}{85{,}000.00} = 9.25926.$$

Furthermore,

$$\bar{c}_A = \frac{P_A''(.08)}{P_A(.08)} = \frac{16{,}675{,}026}{85{,}000} = 196.17677$$

and

$$\bar{c}_L = \frac{P_L''(.08)}{P_L(.08)} = \frac{8{,}016{,}118}{85{,}000} = 94.30727.$$

The immunization strategy is successful because (1) $P_A(.08) = P_L(.08)$, (2) $\bar{v}_A = \bar{v}_L$, and (3) $\bar{c}_A > \bar{c}_L$, as required.

35. (a) To exactly match the obligations, we need total bond payments of 10,000 at each of $t=1$ and $t=2$. Since each bond has a yield rate of 5%, the total present value (which is the total cost) is

$$X = 10,000a_{\overline{2}|.05} = 18,594.$$

(b) This is a more meaningful question than that posed in part (a). The 4% bond pays a coupon and its maturity value at $t=1$. If the face amount of this bond is F_1, then the total payment is $1.04F_1$.

The 6% bond pays a coupon at $t=1$ and a coupon plus maturity value at $t=2$. If the face amount of this bond is F_2, then the payment at $t=1$ is $.06F_2$ and the payment at $t=2$ is $1.06F_2$. F_1 and F_2 must satisfy the equations

$$1.06F_2 = 10,000$$

and

$$.06F_2 + 1.04F_1 = 10,000$$

from which we find $F_1 = 9071.12$ and $F_2 = 9433.96$.